2 Dimensional System of 1st-Order Differential Equations

Example:

$x' = -2y$
$y' = \frac{1}{2} x$
Both x(t) and y(t) are functions with the variable t.
We want to solve this system for both the function x and the function y.

Since the system that we are given is a first order system already, we do not need to transform it into a system of 1st order differential equations using the previous method. It is called 2-dimensional because we have started off with 2 functions x and y.

Step 1: get a differential equation which only has x's and no y's in it, and solve for x

$$x'' = -2y' = -2(\tfrac{1}{2}x) = -x$$
$$x'' + x = 0$$
$$\lambda^2 + \lambda = 0, \qquad \lambda = \pm i$$
$$x(t) = c_1 \sin t + c_2 \cos t$$

Since c_1 and c_2 are arbitrary constants, we can in fact determine them by

$$c_1 = C \cdot \sin\alpha$$
$$c_2 = C \cdot \cos\alpha$$

where C is some constant and α is some angle.

$$x(t) = C\sin\alpha\sin t + C\cos\alpha\cos t = C\cos(t-\alpha)$$

Step 2: get a differential equation which only has y's and no x's in it, and solve for y

$$x' = -2y$$
$$y = -\tfrac{1}{2}x'$$
$$x' = -c_1\cos t + c_2\sin t = -C\sin(t-\alpha)$$
$$y(t) = -\tfrac{1}{2}[-c_1\cos t + c_2\sin t] = \tfrac{1}{2}[c_1\cos t - c_2\sin t]$$
$$OR\ \ y(t) = -\tfrac{1}{2}[-C\sin(t-\alpha)] = \tfrac{C}{2}\sin(t-\alpha)$$

Step 3: figure out what the slope field of the system would look like

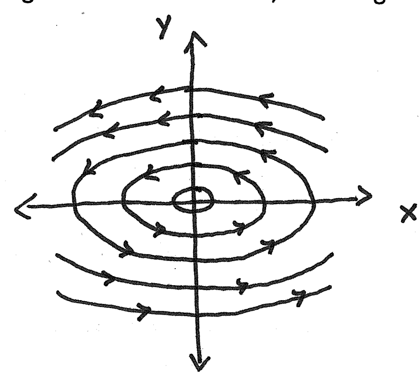
To do this, we need to find some relationship between x and y. Now since they are both trig functions (sin and cos), we can use the following relation: $\sin^2 x + \cos^2 x = 1$

$$\frac{x^2}{C^2} + \frac{y^2}{(C/2)^2} = \frac{C^2\cos^2(t-\alpha)}{C^2} + \frac{(C/2)^2\sin^2(t-\alpha)}{(C/2)^2} = \sin^2(t-\alpha) + \cos^2(t-\alpha) = 1$$

Thus, we get

$$\frac{x^2}{C^2} + \frac{y^2}{(C/2)^2} = 1$$

which is the equation of an ellipse which is twice as wide as it is tall and whose center is the origin. Notice as we pick different C's we will get different solutions, but the general shape of the relationship will be the same.



Suppose now that we want the particular solution of the system at x(0)=2 and y(0)=0

Step 1: solve for c_1 and c_2 in our equations of x and y (this is easier than trying to find C and α)

$$x(t) = c_1 \sin t + c_2 \cos t$$
$$x(0) = c_1 \cdot 0 + c_2 \cdot 1 = c_2 = 2$$

So we have that

$$x(t) = c_1 \sin t + 2 \cos t$$

Next we consider y:

$$y(t) = \frac{c_1}{2}\cos t - \frac{c_2}{2}\sin t$$
$$y(t) = \frac{c_1}{2}\cos t - \sin t$$
$$y(0) = \frac{c_1}{2}\cdot 1 - 0 = \frac{c_1}{2} = 0$$

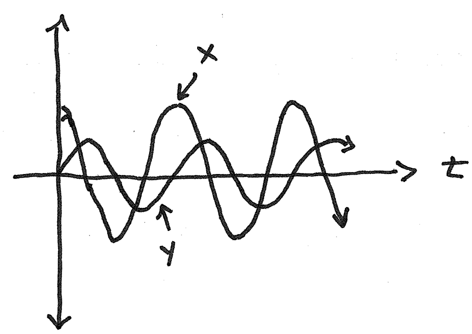
So we get that

$$y(t) = -\sin t$$

And

$$x(t) = 2\cos t$$

Looking at a graph of the x and y function we can see their actual movement:



how functions x and y change over time (t)

If we want to see how the x and y move in relation to each other, would need to find C

C = 2 because

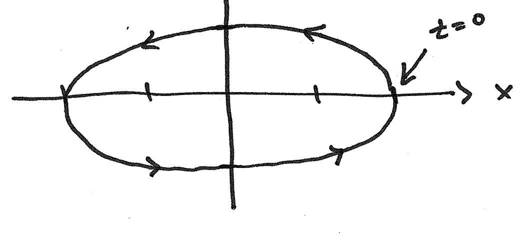
0 = $c_1 = C \cdot \sin\alpha$

2 = $c_2 = C \cdot \cos\alpha$

Then C = 2 and α = 0

Which gives us

$$\frac{x^2}{4} + y^2 = 1$$



how x & y are related "proportional"